Second Order Linear Homogeneous Differential Equations with Constant Coefficients

Henceforth '*second order homogeneous differential equations*'

Definition. A differential equation of the form

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \text{ or } y'' + ay' + by = 0$$

Example

Solve $y' - y = 0$

Method 1:

$$\begin{aligned} & \frac{dy}{dx} = y \\ \Rightarrow \quad & \int \frac{1}{y}\,dy = \int dx \\ \Rightarrow \quad & \ln|y| = x + C \\ \Rightarrow \quad & y = Ae^x \end{aligned}$$

Method 2:

$$\begin{aligned} & 0 = \frac{dy}{dx} - y \\ \Rightarrow \quad & 0 = e^{-x}(y' - y) \\ & = \frac{d}{dx}(e^{-x}y) \\ \Rightarrow \quad & A = e^{-x}y \\ \Rightarrow \quad & y = Ae^x \end{aligned}$$

Example

Solve $y' - cy = 0$

Separating variables: $\frac{dy}{y} = c\,dx$, *so* $\ln|y| = cx + C$, *giving* $y = Ae^{cx}$ *where* $A = \pm e^C$.

Example

Solve $y'' + 3y' + 2y = 0$

Suppose $e^{\lambda x}$ is a solution, then

$$
\begin{aligned}
0 &= \lambda^2 e^{\lambda x} + 3\lambda e^{\lambda x} + 2 = 0 \\
\Rightarrow \quad 0 &= \lambda^2 + 3\lambda + 2 \\
&= (\lambda + 2)(\lambda + 1)
\end{aligned}
$$

So if we considered e^{-2x} or e^{-x} they would both be solutions of this differential equation.

The general solution will be $Ae^{-2x} + Be^{-x}$

Example

Solve $y'' + 4y' + 5y = 0$

Using the same logic as before we end up with an ***auxiliary equation*** *of the form $\lambda^2 + 4\lambda + 5 = 0$ which has roots $\lambda = \frac{-4 \pm \sqrt{16 - 4 \cdot 1 \cdot 5}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$*

Therefore we should expect our solutions to be $Ae^{(-2+i)x} + Be^{(-2-i)x}$

However, we can re-write this as $e^{-2x}(C \cos x + D \sin x)$

Remark (Why try an exponential?)**.** For a linear DE with constant coefficients, we seek a function whose derivatives are proportional to itself. The exponential $y = e^{\lambda x}$ has this property: $y' = \lambda e^{\lambda x} = \lambda y$ and $y'' = \lambda^2 y$, etc. Substituting into $y'' + ay' + by = 0$ gives

$$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = 0$$

Since $e^{\lambda x} \neq 0$, we can divide through to get the **auxiliary equation** $\lambda^2 + a\lambda + b = 0$.

Example

Solve $y'' - 4y' + 4y = 0$, such that $x = 0, y = 1, y' = -1$

The auxiliary equation has a repeated root $\lambda = 2$. This means the solution will be of the form $(A + Bx)e^{2x}$. We can check this works. We also need $A = 1$ and $y = (1 + Bx)e^{2x}, y' = 2e^{2x} + Be^{2x} + 2Bxe^{2x}$ so $y'(0) = 2 + B$ so $B = -3$

and the particular solution is $y = (1 - 3x)e^{2x}$

Fact — To find the solution of the differential equation $y'' + ay' + by = 0$

Step 1: Form the **auxiliary equation** $\lambda^2 + a\lambda + b = 0$

Step 2: Solve the auxiliary equation to find two roots, α, β.

Step 3: **If** $\alpha \neq \beta$ are both real, then the general solution is

$$y = Ae^{\alpha x} + Be^{\beta x}$$

Step 4: **If** $\alpha = p + iq = \beta^*$ are complex conjugate roots, then the general solution is

$$y = e^{px}(A\cos qx + B\sin qx)$$

Step 5: **If** $\alpha = \beta$ is a repeated root then the general solution is

$$y = (A + Bx)e^{\alpha x}$$

Step 6: Apply any *initial* or *boundary* conditions to find the constants and give a **particular solution**

Second Order Linear Non-Homogeneous Differential Equations

Example

Why are these differential equations called **linear**?

Because we can scale them by constants and add them, and still have solutions.

Example

Solve $y'' - 3y' + 2y = 4$

Notice that if we had a solution of the differential equation $y'' - 3y' + 2y = 0$ we could add it to any solution and it would still be valid.

So we try and find one particular integral which works, and then add it to our solution.

One easy to find particular integral here might be $y = 2$

So our final solution will be $y = Ae^{2x} + Be^{x} + 2$

Example

Solve $y'' + 6y' + 8y = x$

First notice our complementary function will be $y = Ae^{-2x} + Be^{-4x}$. Let's try and guess an ***ansatz*** *to solve the specific case. Say $y = px + q$*

$$\begin{aligned} y'' &= 0 \\ y' &= p \\ y &= px + q \\ \Rightarrow \quad x &= 6p + 8(px + q) \\ &= 8px + (8q + 6p) \\ \Rightarrow \quad p &= \frac{1}{8}, q = -\frac{3}{4}p = -\frac{3}{32} \end{aligned}$$

Therefore the general solution is $y = Ae^{-2x} + Be^{-4x} + \frac{1}{8}x + -\frac{3}{32}$

Example

Solve $y'' - 3y' + 2y = e^x$

If we try the same thing, we end up with a problem since e^x is already a solution of our homogeneous de. However, we try the same classic trick, multiply by x, so

$$
\begin{aligned}
& & y &= kxe^x \\
&\Rightarrow & y' &= ke^x + kxe^x \\
&\Rightarrow & y'' &= ke^x + ke^x + kxe^x = (kx+2k)e^x \\
&\Rightarrow & e^x &= (kx+2k)e^x - 3(ke^x + kxe^x) + 2(kxe^x) \\
&\Rightarrow & 1 &= (k - 3k + 2k)x + (2k - 3k) \\
&\Rightarrow & k &= -1
\end{aligned}
$$

Therefore our solution will be $y = Ae^{2x} + Be^x - xe^x$

Example

Solve $y'' - 4y' + 4 = e^{2x}$

Our complementary function will be $y = (Ax+B)e^{2x}$ so clearly we can use e^{2x} or xe^{2x} as an Ansatz. So what should we try? Maybe kx^2e^{2x}

$$
\begin{aligned}
& & y &= kx^2e^{2x} \\
& & y' &= 2kxe^{2x} + 2kx^2e^{2x} \\
& & y'' &= 2ke^{2x} + 4kxe^{2x} + 4kxe^{2x} + 4kx^2e^{2x} \\
& & &= e^{2x}(2k + 8kx + 4kx^2) \\
&\Rightarrow & e^{2x} &= e^{2x}(2k + 8kx + 4kx^2) - 4(2kxe^{2x} + 2kx^2e^{2x}) + 4kx^2e^{2x} \\
&\Rightarrow & 1 &= (4k - 8k + 4k)x^2 + (8k - 8k)x + (2k) \\
&\Rightarrow & k &= \frac{1}{2}
\end{aligned}
$$

Therefore our general solution will be $(\frac{1}{2}x^2 + Ax + B)e^{2x}$

Fact — To find the solution of the differential equation $y'' + ay' + by = f(x)$

Step 1: Use the earlier method to solve the homogeneous equation $y'' + ay' + by = 0$. This will give us our **complementary function**

Step 2: Find a suitable **Ansatz** for our $f(x)$.

$f(x)$	Ansatz
c	some constant
x	$px + q$
$p(x)$	some polynomial of the same degree
e^{mx}	ke^{mx}
$\sin mx$ / $\cos mx$	$p\sin mx + q\cos mx$
$x^k \sin mx$	$p(x)\sin mx + q(x)\cos mx$ where p, q have degree k

If our ansatz is a solution to to the differential equation, multiply it by x (and if that's a solution, by x^2 and so on...)

Step 3: Find constants such that the ansatz gives a **particular integral**

Step 4: Add the particular integral to the complementary function to find the **general solution**

Step 5: Apply any *initial* or *boundary* conditions to find the constants and give a **particular solution**

Simple Harmonic Motion (SHM)

Example

A particle, of mass 5 kg, is held on a smooth surface, 0.3 m from a fixed point P, which it is attached to by a spring of natural length 0.2 m and modulus of elasticity 40 N. Describe the motion of the particle once it is released.

$$\begin{aligned} N2(\uparrow): \quad & T = ma \\ \Rightarrow \quad & \frac{\lambda x}{l} = m\ddot{x} \\ \Rightarrow \quad & 0 = \ddot{x} + 40x \end{aligned}$$

This is a differential equation which we can solve, in particular $x(t) = A\cos(\sqrt{40}t) + B\sin(\sqrt{40}t)$.
When $t = 0, x = 0.1, \dot{x} = 0$, *so* $A = 0.1, B = 0$, *therefore the particle follows* $x(t) = 0.1\cos(\sqrt{40}t)$

Fact — Simple Harmonic Motion is modelled by $\ddot{x} = -\omega^2 x$.
Solutions will take the form

$$x(t) = A\sin(\omega t) + B\cos(\omega t) = R\sin(\omega t + \phi)$$

Definition. $\omega = 2\pi f$ is the **angular frequency**

Definition. $T = \frac{2\pi}{\omega} = \frac{1}{f}$ is the **period** (the time it takes to repeat one cycle).

Definition. R is the **amplitude** of the motion. (The maximum displacement from the **equilibrium position**)

Definition. ϕ is the **phase** of the motion

Fact (Energy in SHM) — For a particle of mass m undergoing SHM with $x = R\sin(\omega t + \phi)$:

$$\begin{aligned} \text{Kinetic energy:} \quad & KE = \tfrac{1}{2}m\dot{x}^2 = \tfrac{1}{2}m\omega^2 R^2\cos^2(\omega t + \phi) \\ \text{Potential energy:} \quad & PE = \tfrac{1}{2}m\omega^2 x^2 = \tfrac{1}{2}m\omega^2 R^2\sin^2(\omega t + \phi) \\ \text{Total energy:} \quad & E = KE + PE = \tfrac{1}{2}m\omega^2 R^2 = \text{constant} \end{aligned}$$

Energy oscillates between kinetic and potential, but total mechanical energy is conserved. Maximum speed $v_{\max} = \omega R$ occurs at $x = 0$; maximum displacement $x = \pm R$ occurs when $v = 0$.

Example

For SHM in harmonic form, what is the initial displacement? What is the initial speed?

Example

A spring of natural length 10 cm is attached to a hook in the ceiling. A particle of mass 0.5 kg is attached to the other end of the spring. When the extension of the spring from its natural length is xm the tension in the spring has magnitude $100x$N.

Use $g = 10\text{ms}^{-2}$, giving your final answers to an appropriate degree of accuracy.

(a) Show that, in the equilibrium position, the length of the spring is 15 cm.

(b) Show that, if the spring is displaced from the equilibrium, the particle will perform simple harmonic motion and find the time period of oscillations about this equilibrium

(c) The spring is stretch 2 cm from the equilibrium position and then released. Find the maximum speed of the particle.

Damped Harmonic Motion

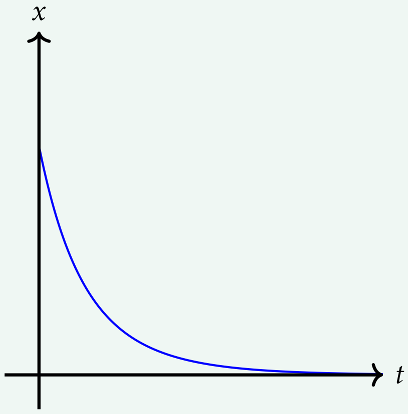
Example

A bob of mass 0.1 kg is connected to a spring. In air, the bob is found to follow SHM with period π seconds. The bob is then placed into oil where there is a drag force of magnitude $0.2v$. Find the motion of the bob.

Fact — If we have an object which would undergo SHM, but for a drag force, we can model it as following a differential equation:

$$\ddot{x} + k\dot{x} + \omega^2 x = 0$$

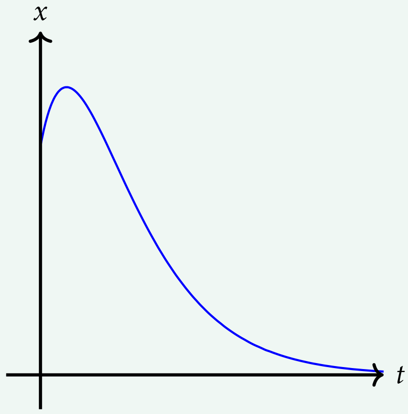
We know from our earlier studies of differential equations that we can solve this differential equation. Looking at the discriminant, $\Delta = k^2 - 4\omega^2$ of the auxiliary equation, we find that:



$k^2 - 4\omega^2 > 0$

Overdamping

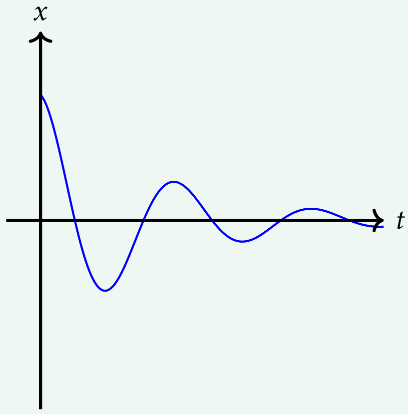
$x = Ae^{\alpha t} + Be^{\beta t}$



$k^2 - 4\omega^2 = 0$

Critical damping

$x = (A + Bt)e^{-\frac{k}{2}t}$



$k^2 - 4\omega^2 < 0$

Underdamping

$x = (A\sin qt + B\cos qt)e^{-\frac{k}{2}t}$

Example (OCR November 2021 - Pure Core 1 Q11)

The displacement of a door from its equilibrium (closed) position is measured by the angle, θ radians, which the door makes with its closed position. The door can swing either side of the equilibrium position so that θ can take positive and negative values. The door is released from rest from an open position at time $t = 0$.

A proposed differential equation to model the motion of the door for $t \geqslant 0$ is

$$\frac{d^2\theta}{dt^2} + \lambda\frac{d\theta}{dt} + 3\theta = 0 \text{ where } \lambda \text{ is a constant and } \lambda \geqslant 0.$$

(a) (i) According to the model, for what value of λ will the motion of the door be simple harmonic? **[1]**

(ii) Explain briefly why modelling the motion of the door as simple harmonic is unlikely to be realistic. **[1]**

(b) Find the range of values of λ for which the model predicts that the door will never pass through the equilibrium position. [2]

(c) Sketch a possible graph of θ against t when λ lies **outside** the range found in part **(b)** but the motion is not simple harmonic. **[1]**

Forced/Driven Harmonic Motion

Example

A particle P of mass 1.5 kg is moving on the x-axis. At time t the displacement of P from the origin O is x metres and the speed of P is v m s^{-1}. Three forces act on P, namely a restoring force of magnitude $7.5x$ N, a resistance to the motion of P of magnitude $6v$ N and a force of magnitude $12\sin t$ N acting in the direction OP. When $t = 0$, $x = 5$ and $\frac{dx}{dt} = 2$.

(a) Show that $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 5x = 8\sin t$.

(b) Find x as a function of t.

(c) Describe the motion when t is large.

Example

A particle P of mass m is attached to one end of a light elastic string AB of natural length l and modulus of elasticity mk^2l. Initially the particle and the string lie at rest on a smooth horizontal plane with $AB = l$. At time $t = 0$ the end B of the spring is set in motion and moves at a constant speed U in the direction AB. The air resistance acting on P has magnitude $2mkv$, where v is the speed of P. At time t the extension of the spring is x and the displacement of P from its initial position is y. Show that, while the string is taut

(a) $x + y = Ut$

(b) $\dfrac{d^2x}{dt^2} + 2k\dfrac{dx}{dt} + k^2x = 2kU$.

(c) Find an expression for x in terms of U, k and t.

Linear systems

Example

A sack containing a liquid chemical is placed in a tank. The chemical seeps out of the sack at a rate of $0.1x$ litres per hour, where x is the number of litres of the chemical remaining in the sack after t hours. The chemical in the tank evaporates at a rate $0.2y$ litres per hour, where y is the number of litres of the chemical in the tank after t hours. If the sack originally contained 50 litres of the chemical, find differential equations for x and for y, and solve them. Find the greatest amount of chemical in the tank, and when this occurs.

Firstly notice that

$$\begin{aligned} & \dot{x} = -0.1x \\ & \dot{y} = 0.1x - 0.2y \\ \Rightarrow \quad & \ddot{y} = 0.1\dot{x} - 0.2\dot{y} \\ & \quad = 0.1(-0.1x) - 0.2\dot{y} \\ & \quad = 0.1(-(\dot{y} + 0.2y)) - 0.2\dot{y} \\ & \quad = -0.3\dot{y} - 0.02y \end{aligned}$$

And this is a linear differential equation we know how to solve.

Of course, real ones solved this back when we did separation of variables...

Example

In a population of foxes (f thousands) and rabbits (r thousands), the foxes have a birth rate $3r$ and a death rate $6f$. The rabbits have a birth rate of $4r$ and a death rate of $8f$.

(a) Write this information in the form of a pair of differential equations.
(b) Rewrite these differential equations as a second order differential equation for f.
(c) Solve this second order differential equation given that initially $f = 2$ and $\frac{\mathrm{d}f}{\mathrm{d}t} = 2$.
(d) Hence find the solution for r, given that the initial population of rabbits is five thousand.
(e) What is the long-term population of foxes and rabbits?